Product of binary relations

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Let $I \neq \emptyset$ be a set of indices, $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ two families of sets. Let us consider the binary relations $\rho_i = (X_i, Y_i, G_i)$, $i \in I$, where G_i is the graph of ρ_i. At the same time we take the cartesian products $X = \prod X_i$ and $Y = \prod Y_i$. We mention that, when I is a finite set, then there exist the cartesian products X and Y, but when I is an infinite set, then we need the axiom of choice to the existence of the cartesian products X and Y.

Definition 1. *We will say that the element $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$ is in the relation ρ with the element $y = (y_i)_{i\in I} \in \prod_{i\in I} Y_i$, denoted by $x\rho y$, if for all $i \in I$ the elements $x_i \in X_i$ are in the relation ρ_i with $y_i \in Y_i$, i.e. $x_i\rho_i y_i$ for every $i \in I$.*

The relation ρ we will call the product of the binary relations $(\rho_i)_{i\in I}$ and we will denote by $\rho = \prod_{i\in I} \rho_i$. We can observe that the graph of the relation ρ is the set $G \subset X \times Y = \left(\prod_{i\in I} X_i\right) \times \left(\prod_{i\in I} Y_i\right)$ given by $G = \{(x, y)/x = (x_i)_{i\in I} \in \prod_{i\in I} X_i = X,\ y = (y_i)_{i\in I} \in \prod_{i\in I} Y_i = Y$ and $x_i\rho_i y_i$ for every $i \in I\}$.

So the product relation is $\rho = (X, Y, G)$.

Proposition 1. *The element $(x, y) \in G$ if and only if $((x_i, y_i))_{i\in I} \in \prod_{i\in I} G_i$.*

Proof. The element $(x, y) \in G$ if and only if $x_i\rho_i y_i$ for all $i \in I$, i.e. $(x_i, y_i) \in G_i$ for every $i \in I$, which means that $((x_i, y_i))_{i\in I} \in \prod_{i\in I} G_i$. $\square$

Let $A \subset X$ be a subset of X.

Definition 2. *The set $\rho(A) = \mathrm{Im}(A) = \{y \in Y/$ there exist $x \in A$ such that $x\rho y\}$ we will call the direct image of the set A by the relation ρ.*

Proposition 2. *The following assertions are true:*

1. *$\rho(\Phi) = Im(\Phi) = \Phi$;*
2. *If $A \subset X$ and $A' \subset X$ are two subsets of X, then:*

   a) $A \subset A'$ implies $\rho(A) \subseteq \rho(A')$;

b) $\rho(A \cup A') = \rho(A) \cup \rho(A')$;

c) $\rho(A \cap A') \subseteq \rho(A) \cap \rho(A')$.

Proof. The proof is standard. $\square$

For every $i \in I$ let us consider the subsets $A_i \subset X_i$ and $A = \prod_{i\in I} A_i \subseteq \prod_{i\in I} X_i = X$.

Proposition 3. *We have* $\rho(A) = \prod_{i\in I} \rho_i(A_i)$.

Proof. The element $y = (y_i)_{i\in I} \in \rho(A)$ iff there exists $x = (x_i)_{i\in I} \in \prod_{i\in I} A_i$ such that $x\rho y$, i.e. for every y_i there exist $x_i \in A_i$ such that $x_i \rho_i y_i$, which means that $y_i \in \rho_i(A_i)$ for every $i \in I$, so $y = (y_i)_{i\in I} \in \prod_{i\in I} \rho_i(A_i)$. $\square$

Let $B \subset Y$ be a subset of Y.

Definition 3. *The set* $\rho^{-1}(B) = \{x \in X \mid \rho(x) \in B\}$ *we will call the inverse image of the set* B *by the relation* ρ.

Proposition 4. *If* $B \subset Y$ *and* $B' \subset Y$ *are two subsets of* Y*, then the following assertions are true:*

1. $B \subseteq B'$ *implies* $\rho^{-1}(B) \subseteq \rho^{-1}(B')$;
2. $\rho^{-1}(B) \cup \rho^{-1}(B') = \rho^{-1}(B \cup B')$;
3. $\rho^{-1}(B \cap B') = \rho^{-1}(B) \cap \rho^{-1}(B')$;
4. $\rho^{-1}(Y) = X$.

Proof. The proof is standard. $\square$

For every $i \in I$ let us consider the subsets $B_i \subset Y_i$ and $B = \prod_{i\in I} B_i \subset \prod_{i\in I} Y_i = Y$.

Proposition 5. *We have* $\rho^{-1}(B) = \prod_{i\in I} \rho_i^{-1}(B_i)$.

Proof. The element $x = (x_i)_{i\in I} \in \rho^{-1}(B)$, iff $\rho(x) \in B = \prod_{i\in I} B_i$, i.e. $\rho_i(x_i) \in B_i$ for every $i \in I$, which means that $x_i \in \rho^{-1}(B_i)$ for every $i \in I$, therefore $x = (x_i)_{i\in I} \in \prod_{i\in I} \rho_i^{-1}(B_i)$. $\square$

Let us consider the sets $(Z_i)_{i\in I}$ and the binary relations $\gamma_i = (Y_i, Z_i, H_i)$, $i \in I$, where H_i are the graph of the binary relations γ_i. Let us denote by γ the product of the binary relations $(\gamma_i)_{i\in I}$, i.e. $\gamma = \prod_{i\in I} \gamma_i$, with the graph $H = \{(y, z) \mid y = (y_i)_{i\in I} \in \prod_{i\in I} Y_i = Y$, $z = (z_i)_{i\in I} \in \prod_{i\in I} Z_i = Z$ and $y_i \gamma_i z_i$ for every $i \in I\}$. So the product relation is $\gamma = (Y, Z, H)$.

Definition 4. *We will call the composition of the relation* $\rho = (X, Y, G)$ *with the relation* $\gamma = (Y, Z, H)$ *the following relation:* $\gamma \circ \rho = (X, Z, H \circ G)$, *where* $H \circ G = \{(x, z)/$ *there exists* $y \in Y$ *such that* $(x, y) \in G$ *and* $(y, z) \in H\}$. *In this case we will say that the element* $x \in X$ *is in the relation* $\gamma \circ \rho$ *with the element* $z \in Z$, *and we denote by* $x\, \gamma \circ \rho\, z$.

Proposition 6. *The element* $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i = X$ *is in the relation* $\gamma \circ \rho$ *with the element* $z = (z_i)_{i \in I} \in \prod_{i \in I} Z_i = Z$ *iff for every* $i \in I$ *the element* $x_i \in X_i$ *is in the relation* $\gamma_i \circ \rho_i$ *with the element* $z_i \in Z_i$.

Proof. We have $x\, \gamma \circ \rho\, z$ if and only if there exists $y \in Y$ such that $x \rho y$ and $y \rho z$. But $x \rho y$ and $y \gamma z$ means that for every $i \in I$ $x_i \rho_i y_i$ and $y_i \gamma_i z_i$, i.e. $x_i \gamma_i \circ \rho_i z_i$ for all $i \in I$. □

Let us consider the sets $(U_i)_{i \in I}$ and the binary relations $\sigma_i = (Z_i, U_i, J_i)$, $i \in I$, where J_i is the graph of the binary relation σ_i. Let us denote by σ the product of the binary relations $(\sigma_i)_{i \in I}$. So $\sigma = \prod_{i \in I} \sigma_i$, i.e. $\sigma = (Z, U, J)$, where $Z = \prod_{i \in I} Z_i$, $U = \prod_{i \in I} U_i$ and $J = \{(z, u) \mid z \sigma u,\ z \in Z$ and $u \in U\}$.

Proposition 7. *Let* $\rho = (X, Y, G)$, $\gamma = (Y, Z, H)$, $\sigma = (Z, U, J)$ *be three relations. Then* $\sigma \circ (\gamma \circ \rho) = (\sigma \circ \gamma) \circ \rho$.

Proof. The proof is standard. □

References

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